Introducing the integrating factor $(x-y)^{-2}$, we have,

$$\frac{(x-y)(2xdx+2ydy)-(x^2+y^2)(dx-dy)}{(x-y)^2}=0,$$

which, upon integration, becomes

$$\frac{x^2+y^2}{x-y}=c, \qquad \text{or} \qquad x^2+y^2-c(x-y)=0.$$

Also solved by A. W. Smith, Norman Anning, J. W. Clawson, J. A. Bullard, G. Paaswell, O. S. Adams, Elijah Swift, Frederick Wood, Horace Olson, C. A. Barnhart, L. M. Coffin, G. W. Hartwell, J. D. Bond, A. G. Rau, C. A. Hutchinson, Claribel Kendall, C. S. Atchinson, J. W. Cromwell, C. P. Sousley, J. A. Caparo, and Paul Capron.

405. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the greatest quadrilateral which can be formed with the four given sides a, b, c, and d taken in order.

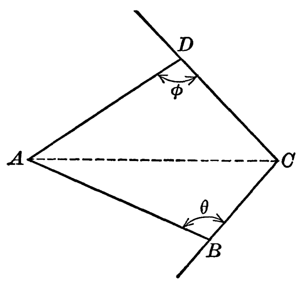
Solution by A. M. Harding, University of Arkansas.

In the quadrilateral $ABCD$, if $AB=a$, $BC=b$, $CD=c$, $AD=d$, we have

$$\overline{AC}^2=a^2+b^2-2ab\cos\theta=c^2+d^2-2cd\cos\phi.$$

Hence,

$$ab\cos\theta-cd\cos\phi=\frac{a^2+b^2-c^2-d^2}{2}. \tag{1}$$



Also, area $=\frac{1}{2}ab\sin\theta+\frac{1}{2}cd\sin\varphi$. For a maximum or minimum we must have

$$ab\cos\theta d\theta+cd\cos\varphi d\varphi=0. \tag{2}$$

From (1) we obtain, by differentiation,

$$-ab\sin\theta d\theta+cd\sin\varphi d\varphi=0. \tag{3}$$

It follows from (2) and (3) that

$$\tan\phi=-\tan\theta; \qquad i.e., \qquad \phi+\theta=180^\circ. \tag{4}$$

Therefore,

$$\cos\theta=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

Hence, the given quadrilateral will be a maximum when it can be inscribed in a circle.

Note.—It is evident from the nature of the problem that it is necessary to consider only *convex* quadrilaterals, and that (4) gives a maximum and not a minimum.

406. Proposed by C. N. SCHMALL, New York City.

Given $f(x+h)+f(x-h)=f(x)\cdot f(h)$, determine by Taylor's theorem or otherwise the nature of the function f.

Solution by W. M. Carruth, Hamilton College, New York.

Given

$$f(x+h)+f(x-h)=f(x)\cdot f(h). \tag{1}$$

Put $h=0$ in this equation. Then $2\cdot f(x)=f(0)\cdot f(x)$. Hence, either

$$f(x)=0, \tag{2}$$

which is a trivial solution of the problem and will be ignored until the end of the discussion, or

$$f(0) = 2. \tag{3}$$

Changing the sign of h in (1) does not alter the left-hand member; hence.

$$f(x) \cdot f(h) = f(x) \cdot f(-h),$$

or $f(h) = f(-h)$. Differentiating this, $f'(h) = -f'(-h)$; whence

$$f'(0) = 0. \tag{4}$$

Differentiate (1) twice with respect to h. Then, $f''(x+h) + f''(x-h) = f(x) \cdot f''(h)$. Putting $h = 0$ in this equation, we get

$$2 \cdot f''(x) = f''(0) \cdot f(x). \tag{5}$$

Replace $f''(0)$, which is a constant, by $\pm 2c^2$; and put y for $f(x)$, and d^2y/dx^2 for $f''(x)$. Although the prime was used to represent differentiation with respect to h, (5) is an identity and the change here to x is legitimate. (5) may now be written

$$\frac{d^2y}{dx^2} \mp c^2 y = 0. \tag{6}$$

Using the upper sign in (6), the solution may be written in the form $y = A \cdot \cosh (cx + B)$, where A and B are constants to be determined. From (3) and (4) above,

$$A \cdot \cosh B = 2, \qquad \text{and} \qquad Ac \cdot \sinh B = 0.$$

Therefore, if $c \neq 0$, we have $B = 0$ and $A = 2$ (the values $B = n\pi i$, $A = (-1)^n \cdot 2$ are no more general than these). Hence, one solution of (1), which is easily verified, is

$$f(x) = 2 \cdot \cosh cx. \tag{7}$$

(The fact that (7), and (8) below, may be verified on substitution in (1) proves that the constant c is an arbitrary constant.)

Using the lower sign in (6), the solution may be written in the form

$$y = A \cdot \cos (cx + B).$$

As before,

$$A \cdot \cos B = 2, \qquad \text{and} \qquad Ac \cdot \sin B = 0:$$

whence

$$B = 0, \qquad \text{and} \qquad A = 2.$$

Another solution of (1) is, therefore,

$$f(x) = 2 \cdot \cos cx. \tag{8}$$

If $c = 0$ in (6), we have the solution $y = Ax + B$. Here, from (3) and (4) as before, $A = 0$, and $B = 2$. But the solution $f(x) = 2$ is included in both (7) and (8), if $c = 0$ is permitted there.

Collecting the results that have been given in (2), (7), and (8), we may say that the function $f(x)$ defined by equation (1) must be equivalent to one of the forms

$$0, \quad 2 \cdot \cosh cx, \quad 2 \cdot \cos cx,$$

where c is any real number. If imaginary values of c are permitted, the last two forms of $f(x)$ are equivalent.

If a solution by Maclaurin's theorem rather than by differential equations is desired, the work following equation (5) above may take this form:

Replace $f''(0)$, which is a constant, by $\pm 2c^2$, and consider the primes to mean differentiation with respect to x. Then (5) may be written

$$f''(x) = \pm c^2 f(x).$$

Successive differentiation of this equation leads to the general result

$$f^{(n+2)}(x) = \pm c^2 f^{(n)}(x),$$

or, if $x = 0$ be substituted,

$$f^{(n+2)}(0) = \pm c^2 f^{(n)}(0). \tag{6'}$$

From (4) and (6′), all the odd ordered derivatives of $f(x)$ are zero when $x = 0$. By successive applications of (6′) we have (since $f''(0) = \pm 2c^2$)

$$f^{\mathrm{IV}}(0) = 2c^4, \quad f^{\mathrm{VI}}(0) = \pm 2c^6, \cdots, f^{(4n)}(0) = 2c^{4n}, \quad f^{(4n+2)}(0) = \pm 2c^{4n+2}.$$

Hence, by Maclaurin's theorem,

$$f(x) = 2 \pm 2c^2x^2/2! + 2c^4x^4/4! \pm 2c^6x^6/6! + \cdots.$$

This gives either

$$f(x) = 2\cdot\cosh cx, \quad \text{or} \quad f(x) = 2\cdot\cos cx,$$

to which solutions $f(x) = 0$ should be added as before.

Also solved by H. C. FEEMSTER, OSCAR S. ADAMS, PAUL CAPRON, and the PROPOSER.

MECHANICS.

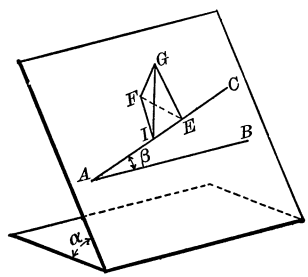
319. Proposed by LAENAS G. WELD, Pullman, Illinois.

A hexagonal pencil lies upon the inclined top of a drawing table and is on the point of either rolling or sliding. Find the angle between its direction and the horizontal edge of the table, the coefficient of friction being μ.

SOLUTION BY H. S. UHLER, Yale University.

The condition for being on the verge of sliding is expressed by $\mu = \tan\alpha$, where α denotes the angle between the table top and the horizontal and is sometimes called the "limiting angle of repose." The proof of this relation seems superfluous in this place because it is given in practically all text-books which are devoted in part or entirely to elementary mechanics.

In order that the pencil may be on the point of rolling, it is obviously necessary and sufficient that the vertical line through the center of gravity intersect the lateral edge which is at the lowest level. In the diagram let $\overline{AB}$ and $\overline{AC}$ indicate respectively a horizontal line on the table top and the lowest lateral edge of the hexagonal prism. G marks the center of gravity which is assumed to lie in the geometric axis of the pencil. $\overline{GI}$ denotes the vertical through G, that is,



the line of action of the weight of the pencil which intersects the edge $\overline{AC}$ in the point I. $\overline{GE}$ and $\overline{GF}$ are perpendiculars dropped from G upon the edge $\overline{AC}$ and the table top, respectively. In other words, the plane EFG contains a right section of the prism passing through the center of gravity. Without quoting the well-known theorems of elementary geometry we see at once that $\angle EFI = \beta$, $\angle EGF = 30°$, $\angle FGI = \alpha$, $\angle EFG = \angle FEI = \angle GFI = 90°$. Consequently, $\overline{FG}/\overline{EF} = \cot 30° = \sqrt{3}$, $\overline{EF}/\overline{FI} = \cos\beta$, and $\overline{FI}/\overline{FG} = \tan\alpha$. Multiplying these three equations together we obtain $1 = \sqrt{3}\cos\beta\tan\alpha$, but $\tan\alpha = \mu$, hence

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}\mu}\right).$$

Remark.—Since the greatest value of a cosine is unity the least value of μ is equal to $1/\sqrt{3} \doteq 0.57735$ corresponding to $\alpha = 30°$ and $\beta = 0°$, as it should be.

Also solved by W. J. THOME, M. R. BOWERMAN, and GEORGE PAASWELL.